J.K.SHAH CLASSES

SYJC : 2016 – 17 **MATHEMATICS & STATISTICS** **DATE : 11/7/2016**

SOLUTION TO PAPER – 1 – SET 2

Q1.

01.

STEP 1

$\lim_{x \to 1} f(x)$

$= \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$

$= \lim_{x \to 1} \frac{(x - 1)(x - 2)}{x - 1}$ $\quad x - 1 \neq 0$

$= \lim_{x \to 1} x - 2$

$= 1 - 2$

$= -1$

STEP 2 :

f(1) = 3 ……… given

STEP 3 :

$f(1) \neq \lim_{x \to 1} f(x)$

$\therefore$ f is discontinuous at x = 1

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at x = 1 by redefining it as

$f(x) = \frac{x^2 - 3x + 2}{x - 1}$; $x \neq 1$

$= -1$; $x = 1$

02.

SOLUTION :

STEP 1

$\lim_{x \to 1} f(x)$

$= \lim_{x \to 1} \frac{x^{12} - 1}{x - 1}$

$= \lim_{x \to 1} \frac{x^{12} - 1^{12}}{x - 1}$

$= 12(1)^{12 - 1}$

$= 12(1)^{11}$

$= 12$

STEP 2 :

f(1) = k ……… given

STEP 3 :

Since f is continuous at x = 1

$f(1) = \lim_{x \to 1} f(x)$;

k = 12

03.

SOLUTION :

STEP 1

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \frac{\log(1 + 2x)}{x}$

$= \lim_{x \to 0} \frac{1}{x} \log(1 + 2x)$

$= \lim_{x \to 0} \log(1 + 2x)^{\frac{1}{x}}$

$= \lim_{x \to 0} \log(1 + 2x)^{\frac{1}{2x} 2}$

$= \log e2$

$= 2\log e$

$= \quad 2$

STEP 2 :

$f(0) = 2$ ……… given

STEP 3 :

$$f(2) = \lim_{x \to 0} f(x)$$

$\therefore$ f is continuous at $x = 0$

04.

SOLUTION :

STEP 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{\sin^2 5x}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{\sin 5x}{x}\right)^2$$

$$= \lim_{x \to 0} 5 \left(\frac{\sin 5x}{5x}\right)^2$$

$$= (5 \cdot 1)^2$$

$$= 25$$

STEP 2 :

$f(0) = 5$ ……… given

STEP 3 :

$$f(0) \neq \lim_{x \to 0} f(x)$$

$\therefore$ f is discontinuous at $x = 0$

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at $x = 0$ by redefining it as

$$f(x) = \frac{\sin^2 5x}{x^2} \quad ; \quad x \neq 0$$

$$= 25 \quad ; \quad x = 0$$

05.

SOLUTION

$$y = \tan^{-1}(\cot 2x)$$

$$y = \tan^{-1} \tan(\pi/2 - 2x)$$

$$y = \pi/2 - 2x$$

Differentiate wrt x

$$\frac{dy}{dx} = 0 - 2$$

$$\frac{dy}{dx} = -2$$

06.

SOLUTION

Taking log on both sides

$$\text{Log } y = \log x \cdot \log(\sin x)$$

Differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx}\log(\sin x) + \log(\sin x) \cdot \frac{d}{dx}\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{1}{\sin x} \frac{d}{dx}\sin x + \log(\sin x) \cdot \frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{1}{\sin x} \cdot \cos x + \frac{\log(\sin x)}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \log x \cdot \cot x + \frac{\log(\sin x)}{x}$$

$$\frac{dy}{dx} = y\left(\log x \cdot \cot x + \frac{\log(\sin x)}{x}\right)$$

$$\frac{dy}{dx} = \sin x^{\log x}\left(\log x \cdot \cot x + \frac{\log(\sin x)}{x}\right)$$

07.

SOLUTION

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{\quad}\right)$$

$$1 + \tan^2\theta$$

$y = \sin^{-1} \sin 2\theta$

$y = 2\theta$

$y = 2\tan^{-1}x$

$$\frac{dy}{dx} = \frac{2}{1 + x^2}$$

08.

SOLUTION

$u = \log (1 + x^2)$

Diff wrt x

$$\frac{du}{dx} = \frac{1}{1 + x^2} \cdot \frac{d}{dx}(1 + x^2)$$

$$\frac{du}{dx} = \frac{2x}{1 + x^2}$$

$v = \cot^{-1}x$

$$\frac{dv}{dx} = \frac{-1}{1 + x^2}$$

Now

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$= \frac{\frac{2x}{1 + x^2}}{\frac{-1}{1 + x^2}}$$

$= -2x$

Q2 A .

01. **SOLUTION :**

STEP 1

$$\lim_{x\to 1} f(x)$$

$$= \lim_{x \to 1} \frac{3 - \sqrt{2x+7}}{x - 1}$$

$$= \lim_{x \to 1} \frac{3 - \sqrt{2x+7}}{x - 1} \; \frac{3 + \sqrt{2x+7}}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{9 - (2x+7)}{x - 1} \; \frac{1}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{9 - 2x - 7}{x - 1} \; \frac{1}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{2 - 2x}{x - 1} \; \frac{1}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{2(1 - x)}{x - 1} \; \frac{1}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{-2(x - 1)}{x - 1} \; \frac{1}{3 + \sqrt{2x+7}} \qquad x - 1 \neq 0$$

$$= \lim_{x \to 1} \frac{-2}{3 + \sqrt{2x+7}}$$

$$= \lim_{x \to 1} \frac{-2}{3 + \sqrt{2 + 7}}$$

$$= \frac{-2}{3 + 3}$$

$$= \frac{-1}{3}$$

STEP 2 :

$f(1) = -1/3$ ……… given

STEP 3 :

$f(1) = \lim_{x\to 1} f(x)$; f is continuous at $x = 1$

02. **SOLUTION :**

STEP 1

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= \lim_{x\to 0} \frac{-2 \sin \frac{3x + 7x}{2} \cdot \sin \frac{3x - 7x}{2}}{x^2}$$

$= \lim$ $-2 \sin \underline{10x}$. $\sin - 4x$

$$\lim_{x\to 0} \frac{2 \qquad \overline{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{-2\sin 5x \;.\; \sin -2x}{x^2}$$

$$= \lim_{x\to 0} \frac{2\ \sin 5x \;.\; \sin 2x}{x^2}$$

$$= \lim_{x\to 0} 2\ \frac{\sin 5x}{x} \;.\; \frac{\sin 2x}{x}$$

$$= \lim_{x\to 0} 2.5\ \frac{\sin 5x}{5x} \;.2.\; \frac{\sin 2x}{2x}$$

$$= 2.5(1)\ .\ 2\ (1)$$

$$= 20$$

STEP 2 :

f(0) = 10……… given

STEP 3 :

$$f(0) \neq \lim_{x\to 0} f(x)$$

$\therefore$ f is discontinuous at x = 0

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{\cos 3x - \cos 7x}{x^2} \quad ; \quad x \neq 0$$

$$= 20 \quad ; \quad x = 0$$

03.

SOLUTION :

STEP 1

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{12^x - 3^x - 4^x + 1}{x\sin x}$$

$$= \lim_{x\to 0} \frac{(3.4)^x - 3^x - 4^x + 1}{x\sin x}$$

$$= \lim_{x\to 0} \frac{3^x.4^x - 3^x - 4^x + 1}{x.\sin x}$$

$$= \lim_{x\to 0} \frac{3^x(4^x - 1) -1(4^x - 1)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{(3^x - 1).(4^x - 1)}{x\tan x}$$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$= \lim_{x\to 0} \frac{\dfrac{(3^x - 1).(4^x - 1)}{x^2}}{\dfrac{x\ .\sin x}{x^2}}$$

$$= \lim_{x\to 0} \frac{\dfrac{3^x - 1}{x}\ .\ \dfrac{4^x - 1}{x}}{\dfrac{\sin x}{x}}$$

$$= \frac{\log 3\ .\ \log 4}{1}$$

$$= \log 3\ .\ \log 4$$

STEP 2 :

Since the f is continuous at x = 0

$$f(0) = \lim_{x\to 0} f(x)$$

$$f(0) = \log 3\ .\ \log 4$$

Q2 B .

01.

SOLUTION :

STEP 1

$$\lim_{x\to 0+} f(x)$$

$$= \lim_{x\to 0} x^2 + a$$

$$= 0^2 + a = a$$

STEP 2

$$\lim_{x\to 0-} f(x)$$

$$= \lim_{x\to 0} 2\sqrt{x^2+1} + b$$

$$= 2\sqrt{0^2+1} + b$$

$$= 2 + b$$

STEP 3

$$f(0) = 0^2 + a$$

$$= a$$

STEP 4

Since f is continuous at x = 0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$2 + b = a = a$$

$$2 + b = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

STEP 5

$$f(1) = 2$$

$$1^2 + a = 2 \qquad \therefore \quad a = 1$$

Sub in (1)

$$2 + b = 1 \qquad \therefore \quad b = -1$$

02.

SOLUTION :

STEP 1

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{(e^{3x}-1)^2}{x.\log(1+3x)}$$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$= \lim_{x\to 0} \frac{\dfrac{(e^{3x}-1)^2}{x^2}}{\dfrac{x.\log(1+3x)}{x^2}}$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{e^{3x}-1}{x}\right)^2}{\dfrac{\log(1+3x)}{x}}$$

$$= \lim_{x\to 0} \frac{\left(3\,\dfrac{e^{3x}-1}{3x}\right)^2}{\log(1+3x)^{\frac{1}{x}}}$$

$$= \lim_{x\to 0} \frac{\left(3\,\dfrac{e^{3x}-1}{3x}\right)^2}{\log\left((1+3x)^{\frac{1}{3x}}\right)^3}$$

$$= \frac{(3.\log e)^2}{\log e^3}$$

$$= \frac{9}{3.\log e}$$

$$= 3$$

STEP 2 :

$$f(0) = 10 \quad \ldots\ldots\ldots \text{ given}$$

STEP 3 :

$$f(0) \neq \lim_{x\to 0} f(x)$$

$\therefore$ f is discontinuous at x = 0

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{(e^{3x}-1)^2}{x.\log(1+3x)} \quad ; \quad x \neq 0$$

$$= 3 \quad ; \quad x = 0$$

03.

SOLUTION :

STEP 1

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{\cos 2x - \cos 6x}$$

$$e^x + \quad 1 - 2$$

$$= \lim_{x\to 0} \frac{\overline{e^x}}{-2\sin\frac{2x+6x}{2}\cdot\sin\frac{2x-6x}{2}}$$

$$= \lim_{x\to 0} \frac{\frac{(e^x)^2 + 1 - 2.e^x}{e^x}}{-2\sin\frac{8x}{2}\cdot\sin\frac{-4x}{2}}$$

$$= \lim_{x\to 0} \frac{\frac{(e^x-1)^2}{e^x}}{-2\sin 4x\cdot\sin -2x}$$

$$= \lim_{x\to 0} \frac{\frac{(e^x-1)^2}{e^x}}{2\sin 4x\cdot\sin 2x}$$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$= \lim_{x\to 0} \frac{\frac{(e^x-1)^2}{x^2 e^x}}{\frac{2\sin 4x\cdot\sin 2x}{x^2.}}$$

$$= \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)^2 \frac{1}{e^x}}{2\,\frac{\sin 4x}{x.}\cdot\frac{\sin 2x}{x}}$$

$$= \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)^2 \frac{1}{e^x}}{2\,.4\frac{\sin 4x}{4x.}\cdot 2\frac{\sin 2x}{2x}}$$

$$= \frac{(\log e)^2}{2.4.(1).2.(1)}$$

$$= \frac{1}{16}$$

STEP 2 :

$f(0) = \frac{1}{16}$

STEP 3 :

$f(0) \neq \lim_{x\to 0} f(x)$

$\therefore$ f is discontinuous at $x = 0$

STEP 4 :

REMOVABLE DISCONTINUITY

f can be made continuous at $x = 0$ by redefining it as

$$f(x) = \frac{e^x + e^{-x} - 2}{\cos 2x - \cos 6x} \quad ; \quad x \neq 0$$

$$= \frac{1}{16} \quad ; \quad x = 0$$

Q3 A .

01.

SOLUTION :

$x^y = e^x$

Taking log on both sides

$y \,.\, \log x = x \,.\, \log e$

$y \,.\, \log x = x$

$$y = \frac{x}{\log x}$$

Differentiating wrt x

$$\frac{dy}{dx} = \frac{\log x\,.\,\frac{d}{dx}x - x\,.\,\frac{d}{dx}\log x}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x\,.\,1 - x\,.\,\frac{1}{x}}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$$

02.

SOLUTION

$$y = \tan^{-1}\frac{x}{1+20x^2}$$

$$y = \tan^{-1}\left(\frac{5x - 4x}{1 + 5x.4x}\right)$$

$$y = \tan^{-1} 5x - \tan^{-1} 4x$$

$$\frac{dy}{dx} = \frac{1}{1+25x^2}\cdot\frac{d}{dx}(5x) - \frac{1}{1+16x^2}\cdot\frac{d}{dx}(4x)$$

$$\frac{dy}{dx} = \frac{5}{1 + 25x^2} - \frac{4}{1 + 16x^2}$$

03.

SOLUTION

$$y = \tan^{-1}\left(\frac{\frac{6 + 5\tan x}{5}}{\frac{5 - 6\tan x}{5}}\right)$$

$$y = \tan^{-1}\left(\frac{\frac{6}{5} + \tan x}{1 - \frac{6}{5}\tan x}\right)$$

$$y = \tan^{-1}\frac{6}{5} + \tan^{-1}\tan x$$

$$y = \tan^{-1}\frac{6}{5} + x$$

Differentiating wrt x

$$\frac{dy}{dx} = 0 + 1$$

$$\frac{dy}{dx} = 1$$

Q3. B.

01. SOLUTION

$$\frac{x^3 - y^3}{x^3 + y^3} = \text{cosec } a$$

$$\frac{x^3 - y^3}{x^3 + y^3} = m \text{ (say)}$$

$$x^3 - y^3 = m(x^3 + y^3)$$

$$x^3 - y^3 = mx^3 + my^3$$

$$x^3 - mx^3 = my^3 + y^3$$

$$x^3(1 - m) = y^3(m + 1)$$

$$x^3\frac{1 - m}{1 + m} = y^3$$

$$y^3 = x^3\frac{1 - m}{1 + m} \quad \text{……… (1)}$$

Differentiating wrt x

$$3y^2\frac{dy}{dx} = \frac{1 - m}{1 + m} . 3x^2$$

$$\frac{dy}{dx} = \frac{1 - m}{1 + m} . \frac{x^2}{y^2}$$

$$\frac{dy}{dx} = \frac{y^3}{x^3} . \frac{x^2}{y^2} \quad \text{…….. from 1}$$

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{…….. PROVED}$$

02.

SOLUTION

$$\mathbf{x = \frac{4t}{1 + t^2}}$$

$$\mathbf{x = 2\ \frac{2t}{1 + t^2}}$$

Put $t = \tan\theta$

$$x = 2\ \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$x = 2\sin 2\theta \quad \text{………. (1)}$$

diff wrt ‘θ’

$$\frac{dx}{d\theta} = 2\cos 2\theta\ .\frac{d\,2\theta}{d\theta}$$

$$= 2\cos 2\theta\ .\ 2$$

$$= 4\cos 2\theta$$

$$\mathbf{y = 3\ \frac{1 - t^2}{1 + t^2}}$$

Put $t = \tan\theta$

$$y = 3\ \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$y = 3\cos 2\theta \quad \text{………. (2)}$$

diff wrt ‘θ’

$$\frac{dy}{d\theta} = 3.\ -\sin 2\theta\ .\frac{d\,2\theta}{d\theta}$$

$$= -3\sin 2\theta\ .\ 2$$

$$= -6\sin 2\theta$$

Now

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{-6\sin 2\theta}{4\cos 2\theta}$$

$$\frac{dy}{dx} = -\frac{3 \sin 2\theta}{2 \cos 2\theta}$$

$$\frac{dy}{dx} = -\frac{\frac{3x}{2}}{\frac{2y}{3}}$$

$$\frac{dy}{dx} = -\frac{9x}{4y}$$

03.

$$y = x^x + x^{\sin x}$$

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dv} \text{ ………..(1)}$$

Now

$$u = x^x$$

Taking log on both sides

$$\text{Log } u = x \,.\, \log x$$

DIFF WRT X

$$\frac{1}{u}\frac{du}{dx} = x \frac{d}{dx}\log x + \log x \frac{d}{dx} x$$

$$\frac{1}{u}\frac{du}{dx} = x \,.\, \frac{1}{x} + \log x \,.\, 1$$

$$\frac{1}{u}\frac{du}{dx} = 1 + \log x$$

$$\frac{du}{dx} = u\,(1 + \log x)$$

$$\frac{du}{dx} = x^x\,(1 + \log x) \text{ ….. (2)}$$

cont.

$$v = x^{\sin x}$$

Taking log on both sides

$$\text{Log } y = \sin x \,.\, \log x$$

DIFF WRT X

$$\frac{1}{v}\frac{dv}{dx} = \sin x \,.\, \frac{d}{dx}\log x + \log x \,.\, \frac{d}{dx}\sin x$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \,.\, \frac{1}{x} + \log x \,.\, \cos x$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{\sin x}{x} + \cos x \,.\, \log x$$

$$\frac{dv}{dx} = v\,\frac{\sin x}{x} + \cos x \,.\, \log x$$

$$\frac{dv}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \,.\, \log x\right) \text{ …(3)}$$

Subs (2) & (3) in (1)

$$\therefore \frac{dy}{dx} = x^x\,(1 + \log x) + x^{\sin x}\left(\frac{\sin x}{x} + \cos x \,.\, \log x\right)$$